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Elementary particles, holography and the BMS group

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In the context of asymptotically flat space-times, it has been suggested to label elementary particles as unitary irreducible representations of the BMS group. We analyse this idea in the spirit of the holographic principle advocating the use of this definition.

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The concept of elementary particle plays a central role in the physical interpretation of quantum field theory; nonetheless we still lack a concrete and universally accepted definition whenever gravity is included and, thus, a non trivial background space-time is considered. The aim of this letter is to advocate that, in the framework of four-dimensional asymptotically flat space-times, a solution to this deficiency exists if the overall problem is set in the context of finding an holographic description for a quantum field theory in such class of space-times, [1].

As a starting point, let us remember that, thought as [2, 3], a system is "elementary" when its Hilbert space carries a single irreducible representation (irrep.) of (the double cover of) the full Poincaré group $P = SL(2,\mathbb{C}) \ltimes T^4$, the semidirect product between the four dimensional translations T^4 and $SL(2,\mathbb{C})$, the group of conformal motions of the 2-sphere S^2. Since an elementary system may admit under internal probing a more complex structure, an *elementary particle* is defined as an elementary system whose states cannot be physically connected to states of another system. Albeit natural, the above definition is unsatisfactory for several reasons [3, 4]: 1) in a Poincaré invariant theory, the mass operator admits only a continuous spectrum whereas observations show only a discrete spectrum of (rest) masses which cannot be described by any finite dimensional Lie group [5], 2) by means of the Wigner approach, it is possible to construct all the kinematical and the dynamical data of a Poincaré free field theory, but massless particles may be labelled either by discrete spins with a finite number of polarizations (unfaithful representations) either by continuous spins with an infinite number of polarizations (faithful representations); only the former have been experimentally observed though there is no theoretical reason to prefer any of the above two choices, 3) more importantly, the definition of elementary particles assumes the flatness of the background discarding any gravitational effect. In a general relativity framework, even in presence of a weak gravitational field, this is not a reasonable request: Poincaré invariance is assumed on the basis that the underlying manifold is maximally symmetric i.e., it is Minkowski whereas, according to Einstein's theory, the degree of symmetry of any other bulk space-time is smaller.

A candidate solution for the above pathologies can be formulated in the context of asymptotically flat (AF) space-times where a natural and universal counterpart for the P group exists [4]. In detail, all AF manifolds share a common boundary structure at past and future null infinity. In a Bondi reference frame ($u = t-r, r, \theta, \varphi$), these submanifolds, $\Im^{\pm}$, topologically equivalent to $S^2 \times \mathbb{R}$, can be endowed with a degenerate metric

$$ds^2 = 0 \cdot du^2 + d\theta^2 + \sin^2\theta d\varphi^2,$$

whose group of diffeomorphisms is the so-called Bondi-Metzner-Sachs group (BMS) which, up to a stereographic projection sending (θ,φ) in $(z,\bar{z})$, is

$$u \to u' = K(z,\bar{z})[u + \alpha(z,\bar{z})], \tag{1}$$

$$z \to z' = \frac{az+b}{cz+d} \quad ad - bc = 1 \wedge a,b,c,d \in \mathbb{C}, \tag{2}$$

where $K(z,\bar{z}) = (1+ \mid z \mid^2)^{-1}(\mid az+c \mid^2 + \mid bz+d \mid^2)$ and where $\alpha(z,\bar{z})$ is an arbitrary real scalar function over S^2 [6]. This transformation identifies the BMS as the semidirect product $SL(2,\mathbb{C}) \ltimes N$ where N is the set of α-functions endowed with a suitable topology usually, but not necessary, chosen as $N = L^2(S^2)$ i.e. the collection of square integrable maps over the 2-sphere [7, 8]. The universality of the boundary structure and the dual role of the BMS

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group as diffeomorphism group of $\Im^{\pm}$ and as asymptotic symmetry group of any AF bulk metric naturally suggests to replace the Poincaré group with the BMS as the fundamental group of symmetry; thus, an elementary particle in an AF space-time is defined by means of an elementary system whose Hilbert space carries a unitary irreducible representation of $SL(2,\mathbb{C}) \ltimes L^2(S^2)$. Although this approach experienced an initial success, no further significant progress was achieved in this field after McCarthy analysis of BMS theory of induced representations. In [6], it was pointed out that, besides unitary irrep. related to the observed Poincaré massive and massless fields, a plethora of other elementary particles existed, so far lacking any experimental evidence. As we have anticipated, considering McCarthy seminal work as a starting point, we will nonetheless advocate the effectiveness of the whole approach; in particular the unwanted pathologies dissapear if we interpret the BMS field theory as a boundary theory encoding holographically the information from *any asymptotically flat* space-time.

Let us briefly comment that holography has been introduced in order to solve the apparent paradox information of black holes by means of a second theory living in a codimension one submanifold (usually the boundary) with a density of data not exceeding the Planck density. An explicit realization of these concepts consists on constructing a field theory on the boundary of the chosen space-time invariant under the action of the asymptotic symmetry group; the bulk data are reconstructed starting directly from those associated to the boundary, explaining how they generate their dynamic and how they can reproduce classical space-time. A concrete example is known in an AdS manifold as the AdS/CFT correspondence [9] and, only recently, a similar investigation has begun in the context of AF space-times [10]. In this latter scenario, the aim is to develop a field theory on $\Im^{\pm}$ invariant under a BMS transformation and, consequently, McCarthy analysis can be naturally interpreted as the initial framework where the boundary kinematic data are studied and classified. In particular, adopting notations and nomenclatures as in [6], a *BMS covariant wave function(al)* is defined as a map

$$\psi : L^2(S^2) \longrightarrow \mathbb{C}^n, \tag{3}$$

transforming under a BMS unitary representation, i.e., in a momentum frame and for any $g = (\Lambda, p(\theta,\varphi)) \in BMS$,

$$\left(D^{\lambda}(g)\psi\right)(p') = e^{i<p,\alpha>_{L^2(S^2)}} U_{\lambda}(\Lambda)\psi(\Lambda^{-1}p'), \tag{4}$$

being $U_{\lambda}(\Lambda)$ a unitary representation of $SL(2,\mathbb{C})$ and $e^{i<p,\alpha>_{L^2(S^2)}}$ the character associated to $p(\theta,\varphi)$. In order to describe an elementary particle (or equivalently a free field), the associated wave function should transform under a unitary and *irreducible* representation; the latter can be constructed from the unitary representation of the little (isotropy) groups $L \subset BMS$ and consequently it is possible to introduce an *induced wave function*:

$$\tilde{\psi} : \mathcal{O}_L \sim \frac{SL(2,\mathbb{C})}{L} \longrightarrow \mathbb{C}^m, \;\; m < n \tag{5}$$

transforming under an irrep. of BMS induced from one of L. Thus, according to this setting, the kinematic data for each elementary particle are fully characterized by L and by the associated Casimir invariant, the squared mass m^2 of the free fields. Let us stress that the set of possible little groups includes $SU(2)$ with $m^2 > 0$, Γ, the double cover of $SO(2)$, with $m^2 = 0$ and a plethora of other non-connected isotropy subgroups, the most notables being the series of finite alternate, cyclic and dihedral groups A_n, C_n, D_n with $n > 2$. The connected little groups provide exactly the unitary irrep. giving rise to the observed Poincaré spins; as a direct consequence, the arbitrariness in the choice of the irrep. associated to massless particles disappears since the faithful one-dimensional representation proper of the BMS Γ little group is fully equivalent to its Poincaré counterpart induced from the two dimensional euclidean group $E(2) \subset P$.

The main handicap emerging from the analysis of the kinematic data is the total absence of an interpretation for the additional "non-Poincaré" degrees of freedom. The paradigm we propose is the following: if a BMS field theory encodes the data from *all* AF manifolds, an elementary particle, living in a fixed background, such as, for example, Minkowski space-time, is described only by means of those boundary degrees of freedom allowing a proper reconstruction of the chosen bulk manifold.

In order to support such conjecture, the first step is to compare the dynamic of bulk and boundary free fields. In the context of Wigner approach, the latter can be fully characterized as a set of constraints restricting the covariant wave function to the induced one; in particular, in a BMS setting, starting from (3), these constraints are twofolds: the first is

$$\rho(p)\psi(p) = \psi(p), \tag{6}$$

an orthoprojection equation where $\rho(p)$ is a suitable non a priori invertible covariant operator which cancels the redundant component of $\psi(p)$ in $\mathbb{C}^n$ i.e. the image of f is (isomorphic to) $\mathbb{C}^m$. The second is an orbit constraint that

reduces the support of (3) from $L^2(S^2)$ to the coset space $\mathcal{O}_L$; although an explicit expression is available for all little groups [10], we switch for sake of clarity to a specific example: $L = SU(2)$ where the orbit equation is

$$[\eta^{\mu\nu}\pi(p)_\mu \pi(p)_\nu - m^2]\psi(p) = 0, \quad [p - \pi(p)]\psi(p) = 0, \tag{7}$$

being $\pi(p)$ the so-called Poincaré momentum i.e. a vector constructed by the first four coefficients in the spherical harmonic expansion of each $p(\theta, \varphi)$

$$\pi(p)_\mu = \pi\left(\sum_{l=0}^{\infty}\sum_{l=-m}^{m} p_{lm}Y_{lm}(\theta,\varphi)\right) = (p_{00}, ..., p_{11}).$$

Let us emphasize that, while (7) is the BMS-equivalent of the Klein-Gordon equation which holds for any Poincaré covariant elementary particle, (6) is a compact expression for the wave equations of any BMS free field i.e. they are the BMS counterpart for usual formulas such as as the Dirac and the Proca wave equations. Thus, following this line of reasoning, the pair $\left\{D^\lambda(\Lambda), \rho(p)\right\}$ (from (4) and (6)) completely characterise the dynamic of a free field; each (BMS) elementary particle is distinguished from another only by the values of the squared mass and of the spin.
Since, according to the holographic principle, the boundary theory should encode the bulk degrees of freedom, a comparison, between the classical dynamic of a theory living on $\Im^\pm$ and of one living on a flat background, should be performed at a level of phase spaces. The subtlety lies in the intrinsic infinite dimensional nature of the BMS field theory which prevents a canonical approach to the construction of the phase space since the usual splitting of a four dimensional manifold M^4 as $\Sigma_3 \times \mathbb{R}$ is meaningless in the boundary framework. Thus we introduce the *covariant phase space*, the set of covariant wave function(al)s satisfying the equations of motion and, consequently, representing the dynamically allowed configurations; in the specific example of a BMS $SU(2)$ field it is

$$\Gamma^{(cov)}_{BMS} = \left\{\psi : L^2(S^2) \to \mathbb{R}, [\pi(p)^\nu \pi(p)_\nu - m^2]\psi(p) = 0, [p - \pi(p)]\psi(p) = 0, \quad \rho(p)\psi(p) = \psi(p)\right\}. \tag{8}$$

The Poincaré counterpart of this expression for an $SU(2)$ field is:

$$\Gamma^{(cov)}_{P} = \left\{\psi : T^4 \to \mathbb{R}, [p^\mu p_\mu - m^2]\psi(p) = 0, \ \ \rho(p^\mu)\psi(p^\mu) = \psi(p^\mu)\right\}. \tag{9}$$

It is straightforward to realize that, due to the orbit constraint $[p - \pi(p)]\psi(p) = 0$, (8) is in 1:1 correspondence with (9); furthermore, an identical claim holds between the covariant phase space of a Poincaré $E(2)$ massless field and a BMS Γ massless particle with vanishing pure supertranslational component [11]. In an "holographic" language, this result grants us that the boundary theory fully encodes the bulk classical degrees of freedom (at least in Minkowski); conversely, from an "elementary particle" point of view, the results from [4] are considerably improved since, not only the kinematic but also the *dynamic* of massive and massless Poincaré elementary particles is fully reproduced in a BMS invariant theory.

As a final step, we need to provide evidences that all other BMS little group do not encode any information allowing a full reconstruction of the physics and the geometry of a Minkowski space-time. A solution to this obstacle lies in the so called null surface formulation of general relativity. In this approach to Einstein's theory, the main variable is a scalar function $Z : M^4 \times S^2 \to \mathbb{R}$ (cut function) solution of the light cone equation in M^4 [12]; $Z(x_a, \theta, \varphi)$ allows to univocally reconstruct all the conformal data of the bulk manifold and, in particular, up to a conformal rescaling the metric itself. From an holographic perspective, the appealing aspect of the overall procedure arises realizing that, helding fixed the bulk point $x_a \in M^4$, the cut function is a real scalar map on $\Im^\pm$, thus a boundary data. Moreover since Z_{x_a} is smooth and single-valued in a suitable neighbourhood of $\Im^\pm$, it can be naturally identified as a BMS supertranslation. Thus, in a BMS field theory, the collection of data encoding the free fields dynamic on a fixed background, can be extracted from the degrees of freedom $\{Z_{x_a}\}$ reconstructing a specific manifold in the null surface formalism.

Within this framework, the set of cut functions appears to play a role similar to the Fefferman-Graham construction for an asymptotically AdS space-time (see for example [13, 14]); this latter tool allows for an *algebraic* reconstruction of bulk data starting from boundary ones whereas the counterpart of this approach in an asymptotically flat space-time produces a set of *differential* equations [15]. Conversely, the null surface formulation of general relativity and, more in detail, $Z_{x_a}(z, \bar{z})$ allows in an asymptotically flat space-time a reconstruction of bulk geometry (in particular the metric) starting only from data living on $\Im^+$ (or $\Im^-$) solving a set of algebraic equations. Furthermore also bulk fields on M^4 can be seen as "dependant" only upon boundary data since, starting only from the cut functions, it is possible to construct the following tetrad Θ^i living at null infinity:

$$u = Z(x_a, z, \bar{z}), \ \omega = (1 + |z|^2)\partial Z(x_a, z, \bar{z}), \ \bar{\omega} = (1 + |z|^2)\bar{\partial} Z(x_a, z, \bar{z}), \ R = (1 + |z|^2)^2 \partial\bar{\partial} Z(x_a, z, \bar{z}),$$

which can be (in principle) inverted as $x_a = x_a[\Theta^i, z, \bar{z}]$. Thus each local bulk field $\phi^\lambda : M^4 \to \mathbb{C}^\lambda$ can be now rewritten as a functional of boundary data i.e. $\phi^\lambda(x_a) = \phi^\lambda[\Theta^i, z, \bar{z}]$.

In particular, if we now consider the specific example of a Minkowski background and if we work in a momentum frame, the cut function is unique [16]:

$$Z_{p_a}(\theta,\varphi) = p(\theta,\varphi) = p_a l^a(\theta,\varphi), \tag{10}$$

where $l^a = \{Y_{00}(\theta,\varphi), ..., Y_{11}(\theta,\varphi)\}$. At a classical level, (10) grants us that the momenta encoding the information from a flat manifold automatically satisfy a vanishing pure supertranslational constraint

$$Z_{p_a} - \pi(Z_{p_a}) = 0, \text{ i.e. } p - \pi(p) = 0 \tag{11}$$

Thus a BMS elementary particle can be related to a Poincaré invariant counterpart living in Minkowski only if the equation of motion for the associated covariant wave functional (8) includes (11). For a fixed little group L, the orbit equation imposes to the classical free field an evolution on a finite dimensional manifold embedded in $L^2(S^2)$; the latter is generated by the action of the coset group $\frac{SL(2,\mathbb{C})}{L}$ on a fixed point $\bar{p} \in L^2(S^2)$ such that $L\bar{p} = \bar{p}$. A decomposition in spherical harmonics proves that the most general expression for $\bar{p}$ is [6, 10]

$$\bar{p} = m + \sum_{l>1} \sum_{m=-l}^{l} p_{lm} Y_{lm}(\theta,\varphi), \tag{12}$$

$$\bar{p} = p_0 + p_0 Y_{11}(\theta,\varphi) + \sum_{l>1} \sum_{m=-l}^{l} p_{lm} Y_{lm}(\theta,\varphi), \tag{13}$$

respectively for a massive and a massless field. From the above two formulas, it is straightforward to see that (11) is equivalent to the constraint $p_{lm} = 0$ for any $l > 1$. A detailed analysis, [10, 11], proved that this request may be satisfied only by the connected isotropy subgroups of the BMS group i.e. $SU(2)$, if we consider (12), and Γ if we consider (13); thus, we may conclude that, at least at a classical level, only these two BMS subgroups encode the information from a Minkowski elementary particle, discarding any physical role for the other "pathological" little groups.

In order to better clarify the role of the BMS group in the definition of elementary particles, we need to comment on the quantum aspects of the boundary theory. In the above framework, we focused our attention mainly on the dynamically allowed configurations whereas, if we wish to calculate quantum data, such as correlation functions, by means of path-integral techniques, we should refer to all the kinematically allowed configurations. The latter are a priori different in a Poincaré and in a BMS field theory and it is natural to wonder if the conjectured correspondence holds also at this level. Thus we need to switch to a Lagrangian formalism; if we consider for sake of simplicity a BMS scalar field $\phi(x)$, the equation of motion (7) can be derived minimizing the following action [17]

$$S(\phi) = \int\limits_{L^2(S^2)} d\mu \left\{ \phi(x)[\eta^{\mu\nu} D_{e_\mu} D_{e_\nu} - m^2]\phi(x) + \sum_{i=1}^{\infty} \gamma_i(x) D_{e_i}\phi(x) \right\}, \tag{14}$$

being e_μ an element of the set $\{Y_{00}(\theta,\varphi), ..., Y_{11}(\theta,\varphi)\}$, e_i one of the set $\{Y_{lm}(\theta,\varphi)\}_{l>1}$, D_{e_i} the infinite dimensional directional derivative along e_i and $\gamma_i(x)$ a Lagrange multiplier. The corresponding partition function is

$$S = \int\limits_{\mathcal{C}} d[\phi] e^{iS(\phi)} = const \cdot det[B]^{-\frac{1}{2}}, \tag{15}$$

$$B = \eta^{\mu\nu} D_{e_\mu} D_{e_\nu} + m^2 + \sum_{i=1}^{\infty} \frac{1}{2\zeta_i}(Q_{e_i} - D_{e_i}) D_{e_i}, \tag{16}$$

where ζ_i is an arbitrary real non vanishing number and where Q_{e_i} is the infinite dimensional multiplication operator along the direction e_i. The propagator $G(x_1 - x_2)$ can be calculated as

$$BG(x_1 - x_2) = i\delta(x_1 - x_2). \tag{17}$$

Up to a Fourier transform, (17) satisfies:

$$[\eta^{\mu\nu} p_\mu p_\nu - m^2 + \sum_{i=1}^{\infty}(p_i D_{e_i} - p_i^2)] G\left(p(\theta,\varphi)\right) = i, \tag{18}$$

where p_μ and p_i are the projections of $p(\theta,\varphi)$ respectively along the directions e_μ and $e_i \in L^2(S^2)$. A physical analysis of (18) has been performed in [17], but, in this letter, we wish simply to emphasize the relation of the above formula with the flat counterpart i.e., if we take into account (10) as the set of possible values of $p(\theta,\varphi)$, (18) reduces to

$$G(p) = \frac{i}{\eta^{\mu\nu}p_\mu p_\nu - m^2},$$

which is the 2-point function in a Minkowski background. Thus, this result suggests us that the conjecture to holographically describe Poincaré elementary particles by means of the BMS group should hold also at a quantum level.

To conclude this letter, we wish to emphasize some remarks on the overall approach:

• in a general picture, elementary particles may also be characterized by an additional set of quantum number $\{\sigma\}$ associated to internal degrees of freedom usually described by means of a (gauge) Lie group G. Nonetheless the indices $\{\sigma\}$ act as absolute superselection rules i.e. external interactions can only modify the momentum and the spin projection along a fixed direction. The suggestion in [18] to relate these degrees of freedom to the irrep. of $I = \frac{BMS}{T^4}$, does not seem to hold in an holographic framework; on the contrary the faithful irrep. of I label the so called IR-sectors of gravity [11]. In a few words, the presence of different infrared sectors of the gravitational field is a measure of the arbitrariness induced by the BMS group in the choice of a specific Minkowski space-time describing the underlying geometry of a bulk field approaching $\Im^\pm$. This specific degree of freedom is related to pure supertranslations and, consequently, to the I group; it represents a direct consequence of the obstruction to reduce the BMS to the Poincaré group, thus it has no reference with internal labels of an elementary particle.

• the absence of a physical interpretation for the non connected little groups disappears in a generic (non stationary) background. Conversely, they may carry information from specific bulk data and an example is provided by the discrete isotropy subgroups, related to gravitational instantons (see [19] and references therein).

• a further question concerns the application of the hypothesis proposed in this letter in a scenario with a non vanishing cosmological constant and in particular in the AdS$_d$/CFT$_{d-1}$ ($d > 3$) correspondence. Let us briefly comment that, though completely different from its asymptotically flat counterpart (formulated only in 4D), in an AdS manifold holography relies to a certain extent on the equivalence between the bulk and the boundary (finite dimensional) symmetry group; thus, a priori, there is no specific reason that do not allows to repeat the reasoning of this letter in such framework though a detailed analysis is not yet available.

• the role of interactions both in bulk and in the boundary has been discarded in this letter since our aim has been to develop an alternative definition of elementary particles which are related to free fields. Nonetheless, in the spirit of finding an holographic correspondence in asymptotically flat space-times, it is imperative to understand the role of interactions between BMS fields and whether they may "break the holographic machinery". According to the initial analysis of the boundary theory in [17], the leading role played by cut functions in the bulk reconstruction starting from boundary data, appears to hold even in presence of interactions. A tricky issue arises if one wishes to consider boundary gauge theory since the usual construction coupling gauge fields and elementary particles, well explained in [20], cannot be blindly applied in the infinite dimensional context proper of a BMS setting; thus this issue is still under analysis and development.

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